



Convergence rates in the limit theorems for random sums of m -orthogonal random variables

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ABSTRACT

In the paper, upper bounds for the convergence rate in the limit theorems for random sums of m -orthogonal random variables are estimated using the K -functional method. Our results are extensions of some known results related to random sums.

1. Introduction

Let $\{X_j, j \geq 1\}$ be a sequence of random variables with the n th partial sum $S_n = \sum_{j=1}^n X_j$, $n \geq 1$, and let N_β be a positive integer-valued random variable whose distribution depends on the positive parameter β , i.e.,

$$\mathbb{P}(N_\beta = n) = p_n = p_n(\beta) \quad (n = 1, 2, \dots),$$

where we assume that $\lim_{\beta \rightarrow +\infty} \mathbb{E}(N_\beta^{-1}) = 0$. Then

$$S_{N_\beta} = \sum_{j=1}^{N_\beta} X_j$$

is called a *random sum*. From now on, it will always be assumed that N_β is independent of the sequence $\{X_j, j \geq 1\}$.

In Robbins (1948), the author studied the random central limit theorem by replacing the index in the partial sum S_n by a random variable N_β . Afterwards, random sums have attracted the attention of researchers, such as Rényi (1956), Klebanov et al. (1985), Klebanov and Rachev (1996), Kalashnikov (1997), Kozubowski and Rachev (1999), Kotz et al. (2001), Chen et al. (2010), Gnedenko and Korolev (2020), Korolev and Zeifman (2021), Hung (2022), and others.

The concept of m -orthogonal sequence was introduced by Petrov (1984). Let m be a non-negative integer. A sequence of random variables $\{X_j, j \geq 1\}$ is said to be *m -orthogonal* if $\mathbb{E}(X_i X_j) = 0$ for all $i \geq 1, j \geq 1$ and $|i - j| > m$.

In particular, a sequence of 0-orthogonal random variables is orthogonal, i.e., $\mathbb{E}(X_i X_j) = 0$ for all $i \neq j$. Obviously, every sequence of mean zero pairwise independent random variables is a 0-orthogonal sequence. However, the reverse is not true. We can also show that the set of all m -orthogonal sequences is really larger than the set of all sequences of mean zero m -dependent random variables and the set of all m -martingale difference sequences. We refer the readers to Hoeffding and Robbins (1948) for the concept of m -dependence, and Thang et al. (2014) for the concept of m -martingale differences.

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This article establishes upper bounds for the convergence rates in the limit theorems for the normalized random sums $\varphi(N_\beta) S_{N_\beta}$ and $\varphi(\mathbb{E}N_\beta) S_{N_\beta}$. Here, the sequence of random variables $\{X_j, j \geq 1\}$ is assumed to be m -orthogonal, and the *normalized function* φ is defined as follows:

$$\varphi : (0, +\infty) \rightarrow (0, +\infty), \quad \varphi(x) = o(1) \quad (x \rightarrow +\infty).$$

It is worth mentioning that the techniques for proving the main results are relatively simple. The obtained results are extensions of some known results related to random sums. An immediate corollary is the rate of convergence in random weak law of large numbers (WLLN) for martingale difference sequences, which was established by Butzer and Schulz (1983). In addition, the WLLN for compound random sums in Hung (2022) and some results of Hung and Hau (2018) will be extended to the m -orthogonal case. We also establish the order of approximation in Rényi's theorem for geometric sums of m -orthogonal random variables.

Throughout the paper, $\xrightarrow{P}$ stands for convergence in probability, $\xrightarrow{d}$ stands for convergence in distribution, $\stackrel{d}{=}$ stands for equality in distribution. All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set of all positive integers will be denoted by $\mathbb{N}$, and the set of all real numbers will be denoted by $\mathbb{R}$.

2. Preliminaries

In the section, we mainly present some definitions and auxiliary lemmas.

Lemma 2.1. *Let m be a non-negative integer, and let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables. Then*

$$\mathbb{E}S_n^2 \leq (m+1) \sum_{j=1}^n \mathbb{E}X_j^2 \quad \text{for all } n \geq 1. \quad (1)$$

Proof. Obviously, the inequality (1) holds for $n \leq m+1$. We now consider the case where $n > m+1$. Then

$$\begin{aligned} \mathbb{E}S_n^2 &= \mathbb{E} \left[\sum_{k=1}^{m+1} \sum_{0 \leq i(m+1) \leq n-k} X_{i(m+1)+k} \right]^2 \leq (m+1) \sum_{k=1}^{m+1} \mathbb{E} \left[\sum_{0 \leq i(m+1) \leq n-k} X_{i(m+1)+k} \right]^2 \\ &= (m+1) \sum_{k=1}^{m+1} \sum_{0 \leq i(m+1) \leq n-k} \mathbb{E} (X_{i(m+1)+k})^2 = (m+1) \sum_{j=1}^n \mathbb{E}X_j^2. \end{aligned}$$

The proof is finished. $\square$

We shall denote by $C(\mathbb{R})$ the class of all bounded, uniformly continuous, real-valued functions f defined on $\mathbb{R}$ with the norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. Let s be a positive integer, we set

$$C^s(\mathbb{R}) = \{f \in C(\mathbb{R}) : f^{(i)} \in C(\mathbb{R}), 1 \leq i \leq s\},$$

where $f^{(i)}$ is the derivative of order i of f .

For any $f \in C(\mathbb{R})$ and $t \geq 0$, we define the K -functional by

$$K_s(t, f) = \inf_{g \in C^s(\mathbb{R})} \left\{ \|f - g\| + t \cdot \|g^{(s)}\| \right\},$$

and the s th modulus of continuity by

$$\omega_s(t, f) = \sup_{0 < |h| \leq t} \left\| \sum_{k=0}^s (-1)^{s-k} \binom{s}{k} f(x + kh) \right\|.$$

The s th modulus of continuity $\omega_s(t, f)$ is a continuous and increasing function of t with $\omega_s(0, f) = 0$ (see DeVore and Lorentz, 1993, page 44 for details).

The proof of the following lemma can be found in DeVore and Lorentz (1993), page 177 (also see Butzer and Berens, 2013, pages 192, 258).

Lemma 2.2. *There exist positive constants $C_{1,s}$ and $C_{2,s}$ which depend only on s such that, for each $f \in C(\mathbb{R})$,*

$$C_{1,s} \omega_s(t, f) \leq K_s(t^s, f) \leq C_{2,s} \omega_s(t, f).$$

For $0 < \alpha \leq s$, we define the Lipschitz class of index $s \in \mathbb{N}$ and order α by

$$\text{Lip}(\alpha, s, L_f) = \{f \in C(\mathbb{R}) : \omega_s(t, f) \leq L_f t^\alpha, t > 0\},$$

where L_f is the Lipschitz constant.

The key technique in the proofs of our main results is provided in the following lemma:

Lemma 2.3. Let U, V be two arbitrary random variables with

$$\mathbb{E}U^i = \mathbb{E}V^i \quad \text{for } i = 1, 2, \dots, s-1.$$

Then there exists a positive constant $C_{2,s}$ which depends only on s such that, for each $f \in C(\mathbb{R})$,

$$|\mathbb{E}f(U) - \mathbb{E}f(V)| \leq 2C_{2,s}\omega_s \left(\left\{ \frac{1}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s) \right\}^{1/s}, f \right). \quad (2)$$

Furthermore, if $f \in \text{Lip}(\alpha, s, L_f)$, then

$$|\mathbb{E}f(U) - \mathbb{E}f(V)| \leq 2^{1-\alpha/s} (s!)^{-\alpha/s} C_{2,s} L_f (\mathbb{E}|U|^s + \mathbb{E}|V|^s)^{\alpha/s}. \quad (3)$$

Proof. For every $f \in C(\mathbb{R})$ and $g \in C^s(\mathbb{R})$, we have

$$\begin{aligned} |\mathbb{E}f(U) - \mathbb{E}f(V)| &\leq |\mathbb{E}f(U) - \mathbb{E}g(U)| + |\mathbb{E}g(U) - \mathbb{E}g(V)| + |\mathbb{E}g(V) - \mathbb{E}f(V)| \\ &\leq 2\|f - g\| + |\mathbb{E}g(U) - \mathbb{E}g(V)|. \end{aligned}$$

Since $g \in C^s(\mathbb{R})$, one has by the Taylor series expansion with Lagrange remainder

$$g(x) = g(0) + \frac{g'(0)}{1!}x + \dots + \frac{g^{(s-1)}(0)}{(s-1)!}x^{s-1} + \frac{g^{(s)}(\theta x)}{s!}x^s$$

for any $x \in \mathbb{R}$ and $\theta \in (0, 1)$. Note that $|g^{(s)}(\theta x)| \leq \|g^{(s)}\|$, and together with the assumption $\mathbb{E}U^i = \mathbb{E}V^i$ for $i = 1, 2, \dots, s-1$, we infer that

$$|\mathbb{E}g(U) - \mathbb{E}g(V)| \leq \frac{1}{s!} \mathbb{E} \left\{ |g^{(s)}(\theta U) U^s| + |g^{(s)}(\theta V) V^s| \right\} \leq \frac{\|g^{(s)}\|}{s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s),$$

hence

$$|\mathbb{E}f(U) - \mathbb{E}f(V)| \leq 2 \left[\|f - g\| + \frac{\|g^{(s)}\|}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s) \right]. \quad (4)$$

Taking the infimum from both sides of (4) over $g \in C^s(\mathbb{R})$, we arrive at

$$|\mathbb{E}f(U) - \mathbb{E}f(V)| \leq 2K_s \left(\frac{1}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s), f \right).$$

By Lemma 2.2, there exists a positive constant $C_{2,s}$ such that

$$K_s \left(\frac{1}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s), f \right) \leq C_{2,s}\omega_s \left(\left\{ \frac{1}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s) \right\}^{1/s}, f \right),$$

this implies that (2) holds. Furthermore, if $f \in \text{Lip}(\alpha, s, L_f)$, then

$$\omega_s \left(\left\{ \frac{1}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s) \right\}^{1/s}, f \right) \leq L_f \left\{ \frac{1}{2s!} (\mathbb{E}|U|^s + \mathbb{E}|V|^s) \right\}^{\alpha/s},$$

and so we obtain (3). $\square$

3. Main results

As in Butzer and Schulz (1983), a random variable Z is said to be φ -decomposable if for each $n \in \mathbb{N}$, there exist independent random variables Z_j , $1 \leq j \leq n$, not depending on n , such that

$$Z \stackrel{d}{=} \varphi(n) \sum_{j=1}^n Z_j.$$

Then for each N_β is independent of the random variables Z_j , $j \in \mathbb{N}$, we have

$$Z \stackrel{d}{=} \varphi(N_\beta) \sum_{j=1}^{N_\beta} Z_j.$$

Indeed, let B be an arbitrary Borel set of $\mathbb{R}$. Then for all $n \geq 1$,

$$\begin{aligned} \mathbb{P}(Z \in B) &= \mathbb{P}\left(\varphi(n) \sum_{j=1}^n Z_j \in B\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\left(\varphi(n) \sum_{j=1}^n Z_j \in B\right) (N_\beta = k)\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\varphi(n) \sum_{j=1}^n Z_j \in B\right) \mathbb{P}(N_\beta = k) \\ &= \sum_{k=1}^{\infty} \mathbb{P}\left(\varphi(k) \sum_{j=1}^k Z_j \in B\right) \mathbb{P}(N_\beta = k) = \sum_{k=1}^{\infty} \mathbb{P}\left(\left(\varphi(k) \sum_{j=1}^k Z_j \in B\right) (N_\beta = k)\right) \\ &= \sum_{k=1}^{\infty} \mathbb{P}\left(\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} Z_j \in B\right) (N_\beta = k)\right) = \mathbb{P}\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} Z_j \in B\right). \end{aligned}$$

The following theorem provides the rate of convergence in the limit theorem for random sums of m -orthogonal random variables, where the limit random variable is φ -decomposable.

Theorem 3.1. Let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables with $\mathbb{E}X_j = \mu_j$ ($j \geq 1$). Let Z be a φ -decomposable random variable such that $\mathbb{E}Z_j = 0$ ($j \geq 1$), and assume that N_β is independent of the random variables Z_j ($j \geq 1$). Then there exists a positive constant C such that, for any $f \in C(\mathbb{R})$,

$$\left| \mathbb{E}f\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j)\right) - \mathbb{E}f(Z) \right| \leq 2C\omega_2 \left\{ \left[\frac{1}{4} \left[(m+1) \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right) + \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}Z_j^2 \right) \right] \right\}^{1/2}, f \right\}.$$

Furthermore, if $f \in \text{Lip}(\alpha, 2, L_f)$, then

$$\left| \mathbb{E}f\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j)\right) - \mathbb{E}f(Z) \right| \leq 2^{1-\alpha} C L_f \left\{ (m+1) \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right) + \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}Z_j^2 \right) \right\}^{\alpha/2}.$$

Proof. We observe that

$$\begin{aligned} \mathbb{E} \left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \right) &= \mathbb{E} \left[\mathbb{E} \left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \middle| N_\beta \right) \right] \\ &= \sum_{n=1}^{\infty} \left\{ \mathbb{P}(N_\beta = n) \mathbb{E} \left(\varphi(n) \sum_{j=1}^n (X_j - \mu_j) \middle| N_\beta = n \right) \right\} = \sum_{n=1}^{\infty} \left\{ p_n \varphi(n) \sum_{j=1}^n \mathbb{E}(X_j - \mu_j) \right\} = 0 \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \right)^2 &= \mathbb{E} \left\{ \mathbb{E} \left[\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \right)^2 \middle| N_\beta \right] \right\} = \sum_{n=1}^{\infty} \left\{ p_n \mathbb{E} \left(\varphi(n) \sum_{j=1}^n (X_j - \mu_j) \right)^2 \right\} \\ &= \sum_{n=1}^{\infty} \left\{ p_n [\varphi(n)]^2 \mathbb{E} \left(\sum_{j=1}^n X_j - \sum_{j=1}^n \mu_j \right)^2 \right\} = \sum_{n=1}^{\infty} \left\{ p_n [\varphi(n)]^2 \mathbb{E} \left[\left(\sum_{j=1}^n X_j \right)^2 - 2 \sum_{j=1}^n X_j \sum_{j=1}^n \mu_j + \left(\sum_{j=1}^n \mu_j \right)^2 \right] \right\} \\ &= \sum_{n=1}^{\infty} \left\{ p_n [\varphi(n)]^2 \left[\mathbb{E} \left(\sum_{j=1}^n X_j \right)^2 - \left(\sum_{j=1}^n \mu_j \right)^2 \right] \right\} \leq \sum_{n=1}^{\infty} \left\{ p_n [\varphi(n)]^2 \mathbb{E} \left(\sum_{j=1}^n X_j \right)^2 \right\} \\ &\leq \sum_{n=1}^{\infty} \left\{ p_n [\varphi(n)]^2 (m+1) \sum_{j=1}^n \mathbb{E}X_j^2 \right\} = (m+1) \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right). \end{aligned}$$

Analogously, one has $\mathbb{E}Z = 0$ and

$$\mathbb{E}Z^2 = \mathbb{E} \left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} Z_j \right)^2 = \sum_{n=1}^{\infty} \left\{ p_n [\varphi(n)]^2 \sum_{j=1}^n \mathbb{E}Z_j^2 \right\} = \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}Z_j^2 \right).$$

Since $\omega_s(t, f)$ is an increasing function of t , applying [Lemma 2.3](#), there exists a positive constant C such that, for any $f \in C(\mathbb{R})$,

$$\begin{aligned} \left| \mathbb{E}f\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j)\right) - \mathbb{E}f(Z) \right| &\leq 2C\omega_2 \left\{ \left[\frac{1}{4} \left[\mathbb{E} \left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \right)^2 + \mathbb{E}Z^2 \right] \right\}^{1/2}, f \right\} \\ &\leq 2C\omega_2 \left\{ \left[\frac{1}{4} \left[(m+1) \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right) + \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}Z_j^2 \right) \right] \right\}^{1/2}, f \right\}. \end{aligned}$$

Moreover, if $f \in \text{Lip}(\alpha, 2, L_f)$, then by [Lemma 2.3](#) again,

$$\left| \mathbb{E}f\left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j)\right) - \mathbb{E}f(Z) \right| \leq 2^{1-\alpha} C L_f \left\{ (m+1) \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right) + \mathbb{E} \left([\varphi(N_\beta)]^2 \sum_{j=1}^{N_\beta} \mathbb{E}Z_j^2 \right) \right\}^{\alpha/2}.$$

This completes the proof. $\square$

The following corollary follows immediately from [Theorem 3.1](#).

Corollary 3.1. Let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables with $\mathbb{E}X_j = \mu_j$ and $\mathbb{E}X_j^2 < +\infty$ for all $j \geq 1$. Let Z be a φ -decomposable random variable such that $\mathbb{E}Z_j = 0$ and $\mathbb{E}Z_j^2 < +\infty$ for all $j \geq 1$, and assume that N_β is independent of the random variables Z_j ($j \geq 1$). Suppose the following conditions are satisfied:

$$\begin{aligned} \mathbb{E} \left[\left[\varphi(N_\beta) \right]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right] &= o(1) \quad \text{as } \beta \rightarrow +\infty; \\ \mathbb{E} \left[\left[\varphi(N_\beta) \right]^2 \sum_{j=1}^{N_\beta} \mathbb{E}Z_j^2 \right] &= o(1) \quad \text{as } \beta \rightarrow +\infty. \end{aligned}$$

Then

$$\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \xrightarrow{d} Z \quad \text{as } \beta \rightarrow +\infty.$$

In addition, when $Z = 0$ almost surely, we get

$$\varphi(N_\beta) \sum_{j=1}^{N_\beta} (X_j - \mu_j) \xrightarrow{P} 0 \quad \text{as } \beta \rightarrow +\infty.$$

Since every martingale difference sequence is 0-orthogonal, [Theorem 3.1](#) implies the following corollary which was proved by [Butzer and Schulz \(1983\)](#). This result provides a random WLLN for martingale difference sequences.

Corollary 3.2. Let $\{X_j, \mathcal{F}_j, j \geq 0\}$ ($X_0 := 0$) be a martingale difference sequence such that $\mathbb{E}X_j^2 < +\infty$ for all $j \geq 1$.

(i) If $f \in \text{Lip}(\alpha, 2, L_f)$, then

$$\left| \mathbb{E}f \left(\varphi(N_\beta) \sum_{j=1}^{N_\beta} X_j \right) - \mathbb{E}f(0) \right| \leq 2^{1-\alpha} C L_f \left[\mathbb{E} \left[\left[\varphi(N_\beta) \right]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right] \right]^{\alpha/2}.$$

(ii) If

$$\mathbb{E} \left[\left[\varphi(N_\beta) \right]^2 \sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right] = o(1) \quad \text{as } \beta \rightarrow +\infty,$$

then

$$\varphi(N_\beta) \sum_{j=1}^{N_\beta} X_j \xrightarrow{P} 0 \quad \text{as } \beta \rightarrow +\infty.$$

In particular, if $\varphi(N_\beta) = N_\beta^{-1}$, then $N_\beta^{-1} \sum_{j=1}^{N_\beta} X_j \xrightarrow{P} 0$ as $\beta \rightarrow +\infty$, i.e., $\{X_j, j \geq 1\}$ satisfies the random WLLN.

The next corollary establishes the convergence rate in the WLLN for the compound random sum. This is an extension of [Theorem 3.1](#) in [Hung \(2022\)](#) to m -orthogonal sequences. As in [Hung \(2022\)](#), let $\{v_j, j \geq 1\}$ be a sequence of independent, identically distributed (i.i.d.) and positive integer-valued random variables with $\mathbb{E}v_1 \in (0, +\infty)$. Let us consider

$$N_n = v_1 + v_2 + \cdots + v_n,$$

and suppose that the random variables $\{v_j, j \geq 1\}$ are independent of $\{X_j, j \geq 1\}$. The random sum

$$S_{N_n} = \sum_{j=1}^{N_n} X_j$$

is called a *compound random sum*.

Corollary 3.3. Let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables with $\mathbb{E}X_j = \mu_j$ for $j \geq 1$, and let $v_1, v_2, \dots$ be positive integer-valued and i.i.d. random variables, independent of the sequence $\{X_j, j \geq 1\}$. Assume that $M = \max_{j \geq 1} \mathbb{E}X_j^2 < +\infty$. If $f \in \text{Lip}(\alpha, 2, L_f)$, then

$$\left| \mathbb{E}f \left(N_n^{-1} \sum_{j=1}^{N_n} (X_j - \mu_j) \right) - \mathbb{E}f(0) \right| \leq 2^{1-\alpha} C L_f (m+1)^{\alpha/2} M^{\alpha/2} n^{-\alpha/2} [\mathbb{E}(v_1^{-1})]^{\alpha/2}, \quad (5)$$

where $N_n = v_1 + v_2 + \cdots + v_n$.

Proof. By using the fact that

$$\frac{1}{a_1 + a_2 + \dots + a_n} \leq n^{-2} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right)$$

for every $a_i > 0$ ($i = 1, 2, \dots, n$), we get

$$\mathbb{E} (N_n^{-1}) \leq n^{-1} \mathbb{E} (v_1^{-1}).$$

Therefore

$$\mathbb{E} \left(N_n^{-2} \sum_{j=1}^{N_n} \mathbb{E} X_j^2 \right) = \sum_{k=1}^{\infty} \left\{ \mathbb{P} (N_n = k) \mathbb{E} \left(k^{-2} \sum_{j=1}^k \mathbb{E} X_j^2 \right) \right\} \leq \sum_{k=1}^{\infty} \{ \mathbb{P} (N_n = k) k^{-1} M \} = M \mathbb{E} (N_n^{-1}) \leq M n^{-1} \mathbb{E} (v_1^{-1}),$$

and from Theorem 3.1, we obtain (5). $\square$

Let $N_{r,p}$ be a negative-binomial random variable with two parameters $r \in \mathbb{N}$ and $p \in (0, 1)$, denoted by $N_{r,p} \sim \mathcal{NB}(r, p)$. In particular, $N_{1,p}$ is a geometric random variable, denoted by $N_{1,p} \sim \text{Geo}(p)$.

Corollary 3.4. Let $N_{r,p} \sim \mathcal{NB}(r, p)$ with $r \in \mathbb{N}$ and $p \in (0, 1)$, and let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables, independent of $N_{r,p}$. Suppose that $\mathbb{E} X_j = \mu_j$ for all $j \geq 1$ and $M = \max_{j \geq 1} \mathbb{E} X_j^2 < +\infty$. If $f \in \text{Lip}(\alpha, 2, L_f)$, then

$$\left| \mathbb{E} f \left(N_{r,p}^{-1} \sum_{j=1}^{N_{r,p}} (X_j - \mu_j) \right) - \mathbb{E} f(0) \right| \leq 2^{1-\alpha} C L_f (m+1)^{\alpha/2} M^{\alpha/2} r^{-\alpha/2} \left[\frac{p}{1-p} \ln(1/p) \right]^{\alpha/2}.$$

Proof. It is well known that $N_{r,p}$ is the sum of r independent geometric random variables with parameter p , i.e.,

$$N_{r,p} = N_{1,p}^{(1)} + \dots + N_{1,p}^{(r)}.$$

We have

$$\mathbb{E} \left(\frac{1}{N_{1,p}^{(1)}} \right) = \sum_{k=1}^{\infty} \frac{1}{k} p (1-p)^{k-1} = \frac{p}{1-p} \sum_{k=1}^{\infty} \frac{(1-p)^k}{k} = \frac{p}{1-p} \ln(1/p),$$

where $\ln(x)$ is the natural logarithm of x . Thus, Corollary 3.3 implies Corollary 3.4. $\square$

Remark 3.1. In Corollary 3.4,

- if p is fixed, then

$$N_{r,p}^{-1} \sum_{j=1}^{N_{r,p}} (X_j - \mu_j) \xrightarrow{P} 0 \quad \text{as } r \rightarrow +\infty;$$

- if r is fixed, then

$$N_{r,p}^{-1} \sum_{j=1}^{N_{r,p}} (X_j - \mu_j) \xrightarrow{P} 0 \quad \text{as } p \rightarrow 0^+.$$

We now say that a random variable Y is φ_e -decomposable with respect to N_β if there exists a sequence of i.i.d. random variables $\{Y_j, j \geq 1\}$ such that

$$Y \stackrel{d}{=} \varphi(\mathbb{E} N_\beta) \sum_{j=1}^{N_\beta} Y_j,$$

where N_β and Y_j ($j \geq 1$) are independent. In the case where $Y, Y_1, Y_2, \dots$ are i.i.d. random variables, and N_β is a geometrically distributed random variable with mean $1/p$ ($0 < p < 1$), then Y is said to be *geometrically strictly stable* (see Klebanov et al. (1985) for more details).

The following examples illustrate the existence of φ_e -decomposable random variables, which will be used in what follows.

Example 3.1. Let $N_p \sim \text{Geo}(p)$ with the probability-generating function

$$\Psi_{N_p}(z) = \frac{pz}{1 - (1-p)z} \quad \text{for } |z| < (1-p)^{-1}.$$

Let $\mathcal{E}$ be an exponential distributed random variable with positive parameter λ , denoted by $\mathcal{E} \sim \text{Exp}(\lambda)$, and let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be independent copies of $\mathcal{E}$, independent of N_p . The characteristic function of $(\mathbb{E} N_p)^{-1} \sum_{j=1}^{N_p} \mathcal{E}_j$ is as follows:

$$\Psi_{N_p}(\psi_{\mathcal{E}_1}(pt)) = \frac{\lambda}{\lambda - it} = \psi_{\mathcal{E}}(t),$$

where $\psi_{\xi}(t)$ denotes the characteristic function of ξ . Therefore

$$\mathcal{E} \stackrel{d}{=} (\mathbb{E}N_p)^{-1} \sum_{j=1}^{N_p} \mathcal{E}_j. \quad (6)$$

Example 3.2. Let $N_{r,p} \sim \mathcal{NB}(r, p)$ with parameters $r \in \mathbb{N}$ and $p \in (0, 1)$. We denote the probability-generating function of $N_{r,p}$ by

$$\Psi_{N_{r,p}}(z) = \left\{ \frac{pz}{1 - (1-p)z} \right\}^r \quad \text{for } |z| < (1-p)^{-1}.$$

Let $\mathcal{G}$ be a gamma distributed random variable with two parameters $r\lambda$ and r , denoted by $\mathcal{G} \sim \text{Gamma}(r\lambda, r)$, where $r \in \mathbb{N}$ and $\lambda > 0$. Let $\{\mathcal{E}_j, j \geq 1\}$ be a sequence of independent and exponentially distributed random variables with parameter λ , independent of $N_{r,p}$. Then the characteristic function of $(\mathbb{E}N_{r,p})^{-1} \sum_{j=1}^{N_{r,p}} \mathcal{E}_j$ will be

$$\Psi_{N_{r,p}}(\psi_{\mathcal{E}_1}((p/r)t)) = \left\{ \frac{p\lambda}{p\lambda - i(p/r)t} \right\}^r = \left\{ \frac{r\lambda}{r\lambda - it} \right\}^r = \Psi_{\mathcal{G}}(t).$$

Thus, we obtain the representation

$$\mathcal{G} \stackrel{d}{=} (\mathbb{E}N_{r,p})^{-1} \sum_{j=1}^{N_{r,p}} \mathcal{E}_j. \quad (7)$$

In the following theorem, we provide a variant of [Theorem 3.1](#) in the φ_e -decomposable case.

Theorem 3.2. Let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables with $\mathbb{E}X_j = \mu$ for all $j \geq 1$, and let Y be a φ_e -decomposable random variable with $\mathbb{E}Y_1 = \mu$ and $\text{Var}(Y_1) = \sigma^2$. Suppose that $\text{Cov}(X_i, Y_j) \geq 0$ for any $i, j \in \mathbb{N}$. Then there exists a positive constant C such that, for any $f \in C(\mathbb{R})$,

$$\left| \mathbb{E}f\left(\varphi(\mathbb{E}N_{\beta}) \sum_{j=1}^{N_{\beta}} X_j - Y\right) - \mathbb{E}f(0) \right| \leq 2C\omega_2 \left\{ \left[\frac{1}{4} \left[(m+1) [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}\left(\sum_{j=1}^{N_{\beta}} \mathbb{E}X_j^2\right) + [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}N_{\beta} \cdot \sigma^2 \right] \right\}^{1/2}, f \right\}. \quad (8)$$

Furthermore, if $f \in \text{Lip}(\alpha, 2, L_f)$, then

$$\left| \mathbb{E}f\left(\varphi(\mathbb{E}N_{\beta}) \sum_{j=1}^{N_{\beta}} X_j - Y\right) - \mathbb{E}f(0) \right| \leq 2^{1-\alpha} C L_f \left\{ (m+1) [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}\left(\sum_{j=1}^{N_{\beta}} \mathbb{E}X_j^2\right) + [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}N_{\beta} \cdot \sigma^2 \right\}^{\alpha/2}. \quad (9)$$

Proof. We have

$$\mathbb{E}\left(\varphi(\mathbb{E}N_{\beta}) \sum_{j=1}^{N_{\beta}} X_j - Y\right) = \varphi(\mathbb{E}N_{\beta}) \left[\mathbb{E}\left(\sum_{j=1}^{N_{\beta}} X_j\right) - \mathbb{E}\left(\sum_{j=1}^{N_{\beta}} Y_j\right) \right] = \varphi(\mathbb{E}N_{\beta}) \sum_{n=1}^{\infty} \left\{ p_n \left[\mathbb{E}\left(\sum_{j=1}^n X_j\right) - \mathbb{E}\left(\sum_{j=1}^n Y_j\right) \right] \right\} = 0$$

and

$$\begin{aligned} \mathbb{E}\left(\varphi(\mathbb{E}N_{\beta}) \sum_{j=1}^{N_{\beta}} X_j - Y\right)^2 &= [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}\left(\sum_{j=1}^{N_{\beta}} X_j - \sum_{j=1}^{N_{\beta}} Y_j\right)^2 = [\varphi(\mathbb{E}N_{\beta})]^2 \sum_{n=1}^{\infty} \left\{ p_n \mathbb{E}\left[\left(\sum_{j=1}^n X_j - n\mu\right) - \left(\sum_{j=1}^n Y_j - n\mu\right)\right]^2 \right\} \\ &= [\varphi(\mathbb{E}N_{\beta})]^2 \sum_{n=1}^{\infty} \left\{ p_n \left[\text{Var}\left(\sum_{j=1}^n X_j\right) + \text{Var}\left(\sum_{j=1}^n Y_j\right) - 2\text{Cov}\left(\sum_{j=1}^n X_j, \sum_{j=1}^n Y_j\right) \right] \right\} \\ &\leq [\varphi(\mathbb{E}N_{\beta})]^2 \sum_{n=1}^{\infty} \left\{ p_n \left[\mathbb{E}\left(\sum_{j=1}^n X_j\right)^2 + n\sigma^2 - 2 \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j) \right] \right\} \\ &\leq [\varphi(\mathbb{E}N_{\beta})]^2 \sum_{n=1}^{\infty} \left\{ p_n \left[(m+1) \sum_{j=1}^n \mathbb{E}X_j^2 + n\sigma^2 \right] \right\} = (m+1) [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}\left(\sum_{j=1}^{N_{\beta}} \mathbb{E}X_j^2\right) + [\varphi(\mathbb{E}N_{\beta})]^2 \mathbb{E}N_{\beta} \cdot \sigma^2. \end{aligned}$$

Applying Lemma 2.3, there exists a positive constant C such that, for any $f \in C(\mathbb{R})$,

$$\begin{aligned} \left| \mathbb{E}f \left(\varphi \left(\mathbb{E}N_\beta \right) \sum_{j=1}^{N_\beta} X_j - Y \right) - \mathbb{E}f(0) \right| &\leq 2C\omega_2 \left\{ \left(\frac{1}{4} \mathbb{E} \left(\varphi \left(\mathbb{E}N_\beta \right) \sum_{j=1}^{N_\beta} X_j - Y \right)^2 \right)^{1/2}, f \right\} \\ &\leq 2C\omega_2 \left\{ \left(\frac{1}{4} \left[(m+1) [\varphi(\mathbb{E}N_\beta)]^2 \mathbb{E} \left(\sum_{j=1}^{N_\beta} \mathbb{E}X_j^2 \right) + [\varphi(\mathbb{E}N_\beta)]^2 \mathbb{E}N_\beta \sigma^2 \right] \right)^{1/2}, f \right\}, \end{aligned}$$

and therefore (8) holds. Moreover, if $f \in \text{Lip}(\alpha, 2, L_f)$, then we obtain (9). $\square$

Remark 3.2. Under the assumptions of Theorem 3.2, if we further assume that $M = \max_{j \geq 1} \mathbb{E}X_j^2 < +\infty$, then, for any $f \in \text{Lip}(\alpha, 2, L_f)$,

$$\left| \mathbb{E}f \left(\varphi \left(\mathbb{E}N_\beta \right) \sum_{j=1}^{N_\beta} X_j - Y \right) - \mathbb{E}f(0) \right| = \mathcal{O} \left(\left\{ [\varphi(\mathbb{E}N_\beta)]^2 \mathbb{E}N_\beta \right\}^{\alpha/2} \right) \quad \text{as } \beta \rightarrow +\infty.$$

Then the condition

$$[\varphi(\mathbb{E}N_\beta)]^2 \mathbb{E}N_\beta = o(1) \quad \text{as } \beta \rightarrow +\infty$$

implies

$$\varphi(\mathbb{E}N_\beta) \sum_{j=1}^{N_\beta} X_j \xrightarrow{P} Y \quad \text{as } \beta \rightarrow +\infty.$$

The following corollary is an extension of Theorems 3.4 in Hung and Hau (2018) to m -orthogonal sequences.

Corollary 3.5. Let $N_{r,p} \sim \mathcal{NB}(r, p)$ with $r \in \mathbb{N}$ and $p \in (0, 1)$, and let $\{X_j, j \geq 1\}$ be a sequence of m -orthogonal random variables, independent of $N_{r,p}$ and $\mathbb{E}X_j = \mu > 0$ for all $j \geq 1$. Let $\{\mathcal{E}_j, j \geq 1\}$ be a sequence of independent and exponentially distributed random variables with parameter $1/\mu$, independent of $N_{r,p}$, such that $\text{Cov}(X_i, \mathcal{E}_j) \geq 0$ for any $i, j \in \mathbb{N}$, and let $\mathcal{G} \sim \text{Gamma}(r\mu^{-1}, r)$. Then for any $f \in C(\mathbb{R})$,

$$\left| \mathbb{E}f \left((p/r) \sum_{j=1}^{N_{r,p}} X_j - \mathcal{G} \right) - \mathbb{E}f(0) \right| \leq 2C\omega_2 \left\{ \left(\frac{1}{4} \left[(m+1)(p/r)^2 \mathbb{E} \left(\sum_{j=1}^{N_{r,p}} \mathbb{E}X_j^2 \right) + (p/r)\mu^2 \right] \right)^{1/2}, f \right\}. \quad (10)$$

If $f \in \text{Lip}(\alpha, 2, L_f)$, then

$$\left| \mathbb{E}f \left((p/r) \sum_{j=1}^{N_{r,p}} X_j - \mathcal{G} \right) - \mathbb{E}f(0) \right| \leq 2^{1-\alpha} C L_f \left\{ (m+1)(p/r)^2 \mathbb{E} \left(\sum_{j=1}^{N_{r,p}} \mathbb{E}X_j^2 \right) + (p/r)\mu^2 \right\}^{\alpha/2}. \quad (11)$$

Proof. By Example 3.2, the gamma random variable $\mathcal{G}$ is decomposed in the form (7). Note that $\mathbb{E}N_{r,p} = r/p$, $\mathbb{E}\mathcal{E}_1 = \mu$ and $\text{Var}(\mathcal{E}_1) = \mu^2$. Therefore, Theorem 3.2 ensures that (10) and (11) hold. $\square$

Data availability

No data was used for the research described in the article.

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